AN IDEAL TOPOLOGY FOR GENERAL BINARY SYSTEMS

by

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TECHNICAL REPORT NO. 10

PREPARED UNDER RESEARCH GRANT NO. NsG-568
(PRINCIPAL INVESTIGATOR: T. N. BHARGAVA)

FOR

NATIONAL AERONAUTICS and SPACE ADMINISTRATION

Submitted for publication to Czechoslovak Mathematical Journal.

DEPARTMENT OF MATHEMATICS

KENT STATE UNIVERSITY

KENT, OHIO

July 1966

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A halfgroupoid is a very general algebraic system consisting of a nonvoid set on which is defined a binary operation, meeting only the requirement of closure, for none or some or all of the possible pairs of elements in the set. In case the operation is defined for all pairs of elements in the set, the system is called a groupoid. The theory of groupoids, which is rather recent, has been investigated by several people, notably Birkhoff [2], Borůvka [3], Bruck [4], and Doyle and Warne [5]. Topological groupoids have been considered mainly by Šulka [9], and Warne [10]. Finally Hanson [6] has studied in his doctoral dissertation connections between binary systems (groupoids) and admissible topologies. However no study of the properties of halfgroupoids seems to have been made; possibly because of the extremely general structure of a halfgroupoid. That these halfgroupoids have some interesting properties was shown by Ohm [8]. In this paper our object is to present a study of an ideal topology associated with halfgroupoids, and to show an interesting connection between such an ideal topology and the digraph topology obtained by Bhargava and Ahlborn [1] in a study of topological properties of directed graphs.

Section 1. Halfgroupoids.

A nonvoid subset S of a halfgroupoid H is called a subhalfgroupoid if and only if $S \cdot S \subset S$. A nonvoid subset A of a halfgroupoid H is called an antihalfgroupoid if and only if $A \cdot A \subset \bar{A}$ where $\bar{A}$ denotes the complement of set A.

A left (right) ideal for a halfgroupoid H is a subset $I_L (I_R) \subset H$

such that $H \cdot I_L(I_R) \subset I_L(I_R)$. A two-sided ideal is a subset $I \subset H$ which is both a left and a right ideal. A point ideal is any ideal consisting of a single element. The empty set is considered a trivial ideal, either left, right, or two-sided.

Let $a,b \in H$; then a is a right factor of b and b is a left multiple of a, denoted by a/b, if there exists an element $c \in H$ such that $c \cdot a = b$. An element $a \in H$ is prime if the set of right factors of a is the empty set or $\{a\}$ itself. An element which is not prime is called composite. An element $a \in H$ for which $a \cdot a = a$ is called an idempotent element. An element $a \in H$ is a right zero element if for all $b \in H$, $b \cdot a = a$. An element $a \in H$ is a right unit element if for all $b \in H$, $b \cdot a = b$.

Definitions corresponding to right ideals or two-sided ideals can be made in an obvious manner but are not given here because throughout this paper we limit ourselves to left ideals only.

Section 2. The L-Topology

Let $\mathcal{I}_L, \mathcal{I}_R, \mathcal{I}_T$ be the families of all left, right, and two-sided ideals respectively for a halfgroupoid H.

THEOREM 2.1 *The family* $\mathcal{I}_L$ *(or* $\mathcal{I}_R$*, or* $\mathcal{I}_T$*) constitutes a topology with the property of completely additive closure.*

Remark 2.1 The family of all possible left, right and two-sided ideals does not, in general, constitute a topology. For example, for the halfgroupoid H defined by

$$H = \begin{array}{c|ccc} & a & b & c \\ \hline a & a & c & a \\ b & c & - & c \\ c & c & b & c \end{array}$$

the family of all possible ideals, left, right or two-sided, is $\{\phi, H, \{a,c\}, \{b,c\}\}$ but $\{a,c\} \cap \{b,c\} = \{c\} \notin \mathcal{I}_L$ or $\mathcal{I}_R$. However $\mathcal{I}_L \cup \mathcal{I}_R$ taken as a subbase does give us a topology.

Remark 2.2 Two halfgroupoids with same elements but different operations may have identical left ideal topologies. For example

$$H_1 = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & - & c & c & d \\ b & c & b & - & - \\ c & - & a & c & a \\ d & b & - & - & b \end{array} \qquad H_2 = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & b & - & c & a \\ b & - & a & c & d \\ c & b & c & - & - \\ d & b & c & - & c \end{array}$$

have exactly same topologies.

Throughout the rest of this paper we call the left ideal topology $\mathcal{I}_L$ as L-topology and denote it simply by $\mathcal{I}$. An element I_L of $\mathcal{I}_L$

is denoted simply as $\mathcal{I}$.

THEOREM 2.2 *The topological space $(H,\mathcal{I})$ is connected if, and only if, there does not exist an I in $\mathcal{I}$ such that $\bar{I}$ is a subhalfgroupoid and $(I\cdot\bar{I})\cap I=\emptyset$.*

THEOREM 2.3 *The topological space $(H,\mathcal{I})$ satisfies T_0 if, and only if, for every a in H there exists a point ideal I_a in $\mathcal{I}$ such that for all b, distinct from a, b is not a right factor of a.*

Proof: Let a belong to H. If a is a right zero element, then $H\cdot\{a\}\subset\{a\}$, so that $\{a\}$ is a point ideal. If a is not a right zero element and there exists some I_a in $\mathcal{I}$ containing no right factor of a and no other element except a, then I_a is a point ideal. If I_a does contain an element $b\neq a$, then $I_a\cap\overline{\{a\}}$ also contains b but not a and is therefore an open set in $\mathcal{I}$. Thus in every case T_0 is satisfied.

Conversely, suppose the condition is not met, i.e. for some right nonzero a in H the point ideals I_a in $\mathcal{I}$ contain at least one more element, other than a, which is a right factor of a. Then $\bigcap\{I_a\}$ contains some element $b\neq a$, where b is such that b/a. Since b/a, there exists an x in H such that $x\cdot b = a$ so that a belongs to $H\cdot I_b$ for any arbitrary I_b in $\mathcal{I}$; but $H\cdot I_b\subset I_b$ so that a belongs to every I_b. Hence $(H,\mathcal{I})$ does not satisfy axiom T_0.

Remark 2.3 For the halfgroupoid

$$H=\begin{array}{c|ccc} & a & b & c \\ \hline a & a & b & c \\ b & a & c & b \\ c & a & b & c \end{array}$$

the topological space $(H,\mathcal{I})$ does not satisfy T_0.

THEOREM 2.4: *The topological space (H,I) satisfies T_1 if, and only if, every element of H is a point ideal; or equivalently if, and only if, every element of H is a right zero element.*

Remark 2.4 Let

$$H_1 = \begin{array}{c|ccc} & a & b & c \\ \hline a & a & b & c \\ b & a & b & b \\ c & a & b & c \end{array}, \quad H_2 = \begin{array}{c|ccc} & a & b & c \\ \hline a & a & b & c \\ b & a & b & c \\ c & a & b & c \end{array}$$

Then (H_1, I) satisfies T_0 but not T_1; and (H_2, I) satisfies T_1.

Remark 2.5 The topological space (H,I) is completely characterized by the conditions of Theorem 2.5, so that all the other separation axioms $T_i (i>1)$ imply T_1.

Section 3. Continuity of Operation.

Following is a rather strong sufficient condition for the binary operation in a halfgroupoid H to be continuous:

THEOREM 3.1 *If every composite element c of a halfgroupoid H is such that every right factor b of c belongs to every point ideal I_c containing c then the binary operation in H is continuous under the L-topology.*

Proof Let a,b,c be any arbitrary elements in H such that $a\cdot b=c$. If $b=c$, then $I_a\cdot I_c \subset I_c$ for every I_a and I_c in $\mathcal{I}$. If $b\neq c$, then c is composite and b belongs to every I_c in $\mathcal{I}$. Thus b belongs to $I_b \cap I_c$ for every I_b, I_c in $\mathcal{I}$. Let $I_b \cap I_c = I_b'$; then $I_a I_b' \subset I_b' \subset I_c$.

Remark 3.1 That the above condition is not necessary is shown by the following example:

$$H = \begin{array}{c|ccc} & a & b & c \\ \hline a & b & - & a \\ b & - & - & b \\ c & a & b & - \end{array} \quad , \quad \mathcal{I} = \{\phi, H, \{b\}, \{a,b\}\}$$

The operation is continuous with respect to L-topology $\mathcal{I}$; and $a\cdot a=b$, i.e. a/b; $\cap\{I_b\}=\{b\}$ so that $a \notin \cap\{I_b\}$.

As a matter of fact something much stronger is true: Let H be a halfgroupoid on k elements for which the L-topology fails to give continuity. Then there exist a,b in H such that $a\cdot b=c$, and $I_a\cdot I_b \not\subset I_c$. Let $H'=H\cup\{x\}, x\varepsilon H$. We define a halfgroupoid on H' by defining the operation in H' to be precisely that of H. Hence all elements of $H\subset H'$ have the same minimal left ideals as in H. Thus for each $k\geq 3$ there exist halfgroupoids for which the minimal left ideals do not yield continuous operations.

Remark 3.2. The continuity of operation in I can be, however, achieved by modifying the original operation in the following way: Let the operation be defined in such a manner that if a,b belong to H, and I_a, I_b are minimal left ideals containing a and b respectively, then $I_a \cdot I_b \subset I_{ab}$, whenever $a \cdot b$ is defined, and $I_{a \cdot b}$ is the minimal left ideal containing $a \cdot b$.

Section 4. A connexion with digraphs.

A digraph (directed graph) $\Gamma(A,E)$ consists of a set A and a subset E of the cartesian product AxA. The points of A are called as vertices and the elements of E as diedges.

DEFINITION 4.1 *The digraph topology T on $\Gamma(A,E)$ is the family of all subsets $B\subset A$ such that $(\bar{B}xB)\cap E=\emptyset$. That is, the set $B\subset A$ is open under T if there are no edges in E which emanate from subset $\bar{B}$ and terminate in subset B. (Bhargava and Ahlborn [1]).*

It has been shown in [1] that the mapping of the set of all possible digraphs, on a fixed set A, onto the set of all digraph topologies is a many-to-one correspondence. This, as we have seen earlier in this paper, is also true of the mapping of the set of all possible halfgroupoids, on a fixed set A, onto the set of all L-topologies.

Let $\mathcal{H}$ be the class of all halfgroupoids on a fixed set A, and let $\mathcal{D}$ be the class of all digraphs on the same set A. Mappings between $\mathcal{H}$ and $\mathcal{D}$ can be constructed such that under these mappings the set $\mathcal{H}(\mathcal{I})=\{H \in \mathcal{H}: \mathcal{I}$ is the L-topology on H$\}$ corresponds to the set $\mathcal{D}(T)=\{D \in \mathcal{D}: T$ is the digraph topology D$\}$, where $\mathcal{I}$ and T are identical topologies.

First of all we note that a binary operation defined for a set A can be considered as a set F of ordered triples, that is $\emptyset\subset F\subset AxAxA$; in terms of our original definition we have $F=\{((a,b),c): a\cdot b=c$ in H$\}$. Let (A,F) denote a halfgroupoid on A with binary operation F, and let (A,E) denote a digraph on set A. Then

$$\mathcal{H}=\{(A,F):\ \emptyset\subset F\subset AxAxA;\ F=\{((a,b),c)\ :\ a\cdot b=c \text{ in } A\}\},$$

and $$\mathcal{D}=\{(A,E):\ \emptyset\subset E\subset AxA\ ;\ E=\{(a,b):\ a,b\in A\}\}.$$

Let $\phi: H \xrightarrow{\text{into}} D$, such that $\phi(H,F)=(H,E)$ where $E=\{(c,b): \{(x,b),c\} \in F\}$

Let $\phi': \mathcal{D} \xrightarrow{\text{into}} H$, such that $\phi'(H,E)=(H,F)$ where $F=\{\{(c,b),c\}: (c,b) \in E\}$.

The following theorem shows that these mappings ϕ and ϕ', both of which are many-to-one, establish, in a certain sense, a topological correspondence between halfgroupoids and digraphs in such a manner that the L-topology is exactly the same as the digraph topology.

THEOREM 4.1 *Let* $\phi: \mathcal{H} \xrightarrow{into} \mathcal{D}$, $\phi': \mathcal{D} \xrightarrow{into} \mathcal{H}$, $\mathcal{H}(I)$, *and* $\mathcal{D}(T)$ *be defined as above. Then* $\phi(\mathcal{H}(I)) = \mathcal{D}(T)$, *and* $\phi'(\mathcal{D}(T)) = \mathcal{H}(I)$.

The proof of this theorem is quite simple and straightforward, but a little lengthy, and hence is omitted. It may be found in Ohm [8], which also contains some other details and relevant results.

Acknowledgements. We are most grateful to Professor P. H. Doyle of Michigan State University for several useful discussions and suggestions which have resulted in improvement of this paper.

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